Systems of equations with a single solution

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Abstract

We classify general systems of polynomial equations with a single solution, or, equivalently, collections of lattice polytopes of minimal positive mixed volume. As a byproduct, this classification provides an algorithm to evaluate the single solution of such a system.

1 Introduction

The mixed volume is the unique symmetric real-valued function MV of n convex bodies in an n-dimensional vector space V such that

$$\mathrm{MV}(A+B, A_2, \ldots, A_n) = \mathrm{MV}(A, A_2, \ldots, A_n) + \mathrm{MV}(B, A_2, \ldots, A_n)$$

in the sense of Minkowski summation $A + B = \{a + b \,|\, a \in A,\, b \in B\}$, and $\mathrm{MV}(A, \ldots, A) =$ (volume of A) in the sense of a given volume form on V. In what follows, V is always of the form $\mathcal{V} \otimes \mathbb{R}$, where $\mathcal{V}$ is an integer lattice, and the volume form is always chosen in such a way that the volume of the torus $V/\mathcal{V}$ is equal to $n!$. For this volume form, the mixed volume of lattice polytopes (i.e. the ones whose vertices are in $\mathcal{V}$) is always integer.

The criterion for a collection of polytopes to have zero mixed volume was established by D. Bernstein and A. Khovanskii (see [6] for the context and e.g. [5, Lemma 1.2], for a proof):

Proposition 1 *The mixed volume of convex bodies is zero if and only if k of these bodies sum up to a body of dimension strictly smaller than k.*

In this paper, we provide the following classification of lattice polytopes collections with minimal positive mixed volume (by induction on n):

Theorem 1 *A collection of n lattice polytopes in V has unit mixed volume if and only if there exists $k > 0$ such that, up to translations, k of the polytopes are faces of the same k-dimensional volume 1 lattice simplex in a k-dimensional rational subspace $U \subset V$, and the images of the other $n - k$ polytopes under the projection $V \to V/U$ have unit mixed volume.*

Here and in what follows, the volume forms in the subspace $U \subset V$ and in the quotient space V/U are induced from the lattices $\mathcal{U} = U \cap \mathcal{V}$ and $\mathcal{V}/\mathcal{U}$ respectively.

The question of classifying lattice polytopes of small mixed volume is particularly motivated by the study of codimension of discriminants (see e.g. [4, Theorem 3.13], or [2] for details).

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Theorem 1 was conjectured in [4, Conjecture 3.16], and the special case of full-dimensional polytopes was proved in [2, Proposition 2.7].

The mixed volume is related to algebra by the Kouchnirenko-Bernstein formula: a system of n polynomial equations of n variables with Newton polytopes $N_1, N_2 \ldots, N_n$ and general coefficients has $\mathrm{MV}(N_1, N_2, \ldots, N_n)$ solutions, see [1]. Thus, Theorem 1 classifies all general systems of polynomial equations with a unique solution. By the general Gröbner basis argument, the solution of such a system admits a rational expression in terms of the coefficients of the system, and Theorem 1 provides an explicit construction for it (by induction on n):

1) upon a certain monomial change of variables, k of the n equations become linear (non-homogeneous) equations of k variables, from which the k variables can be evaluated;

2) after the substitution of the evaluated variables in the other equations, we obtain $n-k$ general equations of $n-k$ variables with a unique solution and proceed to the next group of simultaneously linearizable equations.

The rest of the text is devoted to the proof of Theorem 1. The key ideas are the use of the Aleksandrov–Fenchel inequality

$$\mathrm{MV}(A, B, C, \ldots)^2 \geqslant \mathrm{MV}(A, A, C, \ldots) \cdot \mathrm{MV}(B, B, C, \ldots) \tag{1}$$

and the notion of mixed fiber bodies:

Definition. The mixed fiber body of the convex bodies $A_1, A_2 \ldots, A_{n-k} \subset V$ in a $(k+1)$-dimensional rational subspace $U \subset V$ is the convex body $X \subset U$ such that $\mathrm{MV}(X, B_1, \ldots, B_k) = \mathrm{MV}(A_1, \ldots, A_{n-k}, B_1, \ldots, B_k)$ for all collections of convex bodies $B_1, B_2 \ldots, B_k \subset U$.

The mixed fiber body always exists, is unique up to a translation, and is a lattice polytope if so are $A_1, A_2 \ldots, A_{n-k}$ (see [3] for the proofs and the equivalence to the original definition [7]). The same ideas allow to classify collections of lattice polytopes of larger mixed volume: in a forthcoming paper, we classify the polytopes, whose mixed volume equals two. In such a collection, it is not always possible to find k elements which are faces of a k-dimensional volume 2 polytope.

2 Theorem 1 for essential tuples

Definition. A set of lattice polytopes $A_1, A_2, \ldots, A_k \subset \mathbb{R}^n$ is called *a k-tuple in* $\mathbb{R}^n$. Depending on the context, we treat a tuple A either as an unordered multiset (if its elements order does not matter) or as a map $A\colon \{1, 2, \ldots, n\} \to \Omega$ to the set of lattice polytopes in $\mathbb{R}^n$.

Definition. A k-tuple A in $\mathbb{R}^n$ is said to be *essential*, if for each $l \leq k$ and $j_1 < j_2 < \ldots < j_l$, we have $\dim \sum_{s=1}^{l} A(j_s) \geq \min(l+1, n)$.

See [8] for a relation between this notion and codimension of resultants. We now reformulate Theorem 1 as follows.

Theorem 2 *Once the mixed volume of an essential n-tuple A in $\mathbb{R}^n$ is 1, then all the polytopes $A(i)$, up to translations, are contained in a volume 1 lattice simplex.*

The proof is given in the last section of the text.

Proof of Theorem 1. For essential tuples, Theorem 1 and Theorem 2 are equivalent. If the tuple A is not essential, then it contains a sub-tuple of k polytopes that sum up to a k-dimensional polytope, and the statement of Theorem 1 can be deduced for this tuple by induction on n from the following well known fact. $\square$

Proposition 2 *If the polytopes $A_1, A_2 \dots, A_k$ of an n-tuple $A = \{A_1, A_2, \dots, A_n\}$ in $\mathbb{R}^n$ are contained in a subspace $\mathbb{R}^k \subset \mathbb{R}^n$, then*

$$\mathrm{MV}(A_1, A_2 \dots, A_n) = \mathrm{MV}(A_1, A_2 \dots, A_k) \cdot \mathrm{MV}(B_{k+1}, B_{k+2} \dots, B_n),$$

where B_j is the image of A_j under the projection along $\mathbb{R}^k$.

Theorem 2 can be extended to mixed fiber bodies as follows. Consider a k-tuple A in $\mathbb{R}^n$ and a rational affine subspace $L \subset \mathbb{R}^n$ of dimension $n-k+1$.

Definition. We call the tuple A *essential with respect to* L if, for each full-dimensional $(n-k)$-tuple B in L, the system $A \cup B$ is essential.

Corollary 1 *Assume that a k-tuple A in $\mathbb{R}^n$ with $0 < k < n$ is essential with respect to a subspace $L \subset \mathbb{R}^n$ of dimension $n-k+1$, and the mixed fiber body $\widetilde{A} \subset L$ of the tuple A is contained, up to translations. in a volume 1 lattice full-dimensional simplex δ in L. Then there exists a volume 1 lattice simplex $\Delta \subset \mathbb{R}^n$ adjacent to δ that contains all the polytopes $A(i)$ up to translations.*

Proof. Consider the tuple B consisting of $n-k$ copies of δ. The tuple $A \cup B$ is essential, and the mixed volume of $A \cup B$ equals the mixed volume of $\{\delta\} \cup B$, which equals 1. Applying Theorem 2 to $A \cup B$, we obtain a required volume 1 simplex Δ. $\square$

Note that the condition $k < n$ is essential. In fact, in the case $k = n$, any essential tuple A is essential with respect to each one-dimensional subspace $L \subset \mathbb{R}^n$. Let $A(i) = \Delta$ be a volume 1 lattice full-dimensional simplex in $\mathbb{R}^n$. Given an arbitrary L, the mixed fiber body $\widetilde{A} \subset L$ is a unit segment. However, the simplex Δ is not adjacent to this segment in general.

3 Preliminary statements

For a lattice polytope $X \in \Omega$, we denote its support function by $l_X \colon (\mathbb{Z}^n)^* \to \mathbb{Z}$, so that $l_X(\alpha) = \max_{x\in X} \alpha(x)$, $\alpha \in (\mathbb{Z}^n)^*$. Denote by $\mathcal{Z} \subset (\mathbb{Z}^n)^*$ the set of primitive integer covectors. For a covector $\alpha \in \mathcal{Z}$, we denote by X^α the face $\{x \in X \mid \alpha(x) = l_X(\alpha)\}$ of the polytope X, where the covector α attains its maximal value. In what follows, we use also the following notation. For a k-tuple A in a rational k-dimensional affine subspace $L \subset \mathbb{R}^n$, we denote the mixed volume of A by $\prod_{j=1}^k A(j) = \prod A$, use $\sum A$ for $\sum_{i=1}^k A(i)$, and use $\dim A$ for $\dim \sum A - k$.

Definition. A k-tuple A in $\mathbb{R}^n$ is said to be *linearly independent* if, for each l and $j_1 < j_2 < \dots < j_l$, one has $\dim \sum_{s=1}^l A(j_s) \geq l$.

The following identity (see [1]) often turns out useful for treatment and computation of the mixed volume of an n-tuple A in $\mathbb{R}^n$:

$$\prod A = \sum_{\alpha\in\mathcal{Z}} l_{A(1)}(\alpha) \prod_{j=2}^n A(j)^\alpha. \tag{2}$$

As a consequence of this equation, the following *monotony property* of the mixed volume holds. For a couple of n-tuples A, B in $\mathbb{R}^n$, if $A(j) \subset B(j)$, $j = 1, 2, \dots, n$, then $\prod A \leq \prod B$.

The following two statements provide the inductive step for the proof of Theorem 2.

Lemma 1 *Let an $(n-1)$-tuple A in $\mathbb{R}^n$ be essential and consist of lattice polytopes contained (up to translations) in a volume 1 lattice full-dimensional simplex Δ in $\mathbb{R}^n$.*

1. *For each covector $\alpha \in \mathcal{Z}$ defining a facet Δ^α, the mixed volume $\prod(A(i))^\alpha$ equals 1.*
2. *For a covector $\alpha \in \mathcal{Z}$ such that $\dim \Delta^\alpha < n-1$, we have $\prod(A(i))^\alpha = 0$.*

Proof. 1. Since the polytopes $B(i) = (A(i))^\alpha$, $i = 1, 2, \ldots, n-1$, lie in the volume 1 simplex Δ^α up to translations, we have $\prod B \leq \mathrm{Vol}(\Delta^\alpha) = 1$. We claim that $\prod B \neq 0$. If not, Proposition 1 provides a subtuple $B' = \{B_{j_1}, B_{j_2}, \ldots, B_{j_k}\}$, $j_1 < j_2 \ldots < j_k$, of B, such that $\dim \sum B' < k$. Consider the elements of B' as k nodes of the graph that has an edge between a pair of polytopes if and only if their intersection is not empty. Let $C_1, C_2, \ldots, C_l$ be the subtuples of B' corresponding to the connected components of the graph. Let X_i be the number of vertices of Δ that belong to the union of polytopes forming C_i. We have $\sum X_i - l = \dim \sum B' < k = \sum_{t=1}^{l} c_t$, where $c_t = |C_t|$. Thus there exists t such that $X_t - 1 < c_t$. Let $B_{s_1}, B_{s_2}, \ldots, B_{s_{c_t}}$ be the polytopes forming C_t. Then $\dim \sum_{i=1}^{c_t} A_{s_i} \leq X_t \leq c_t$, which contradicts to the fact that A is essential.

2. Denote the covectors defining the facets of Δ by $\alpha_1, \alpha_2, \ldots \alpha_{n+1}$. Assume there exists a covector $\alpha_0 \in \mathcal{Z}$ different from α_i such that $\prod(A(i))^{\alpha_0} > 0$. Consider any edge (v_0, v_1) of Δ that is not orthogonal to α_0, assume $\alpha_0(v_0) < \alpha_0(v_1)$. Choose an integer affine coordinate system in $\mathbb{Z}^n$ with the origin at v_0. W.l.o.g., assume that $\Delta^{\alpha_{n+1}}$ is the facet of Δ that does not contain v_0. By the identity (2), one has $\prod(A \cup \{\Delta\}) = \sum_{\alpha \in \mathcal{Z}} l_\Delta(\alpha) \prod(A(i))^\alpha \geq l_\Delta(\alpha_{n+1}) \prod(A(i))^{\alpha_{n+1}} + l_\Delta(\alpha_0) \prod(A(i))^{\alpha_0} \geq 2$. On the other hand, the monotony property of mixed volumes implies that $\prod(A \cup \{\Delta\}) \leq \mathrm{Vol}(\Delta) = 1$, which leads to a contradiction. $\square$

Lemma 2 *Let an $(n-1)$-tuple A in $\mathbb{R}^n$ be essential and consist of lattice polytopes contained (up to translations) in a volume 1 lattice full-dimensional simplex Δ in $\mathbb{R}^n$. If the mixed volume $\prod(A \cup \{X\})$ with some lattice polytope X equals 1, then X is also contained in Δ up to a translation.*

Proof. Chose a coordinate system in $\mathbb{R}^n$ such that Δ is formed by the basis vectors. Let $\alpha_1, \alpha_2, \ldots, \alpha_{n+1} \in \mathcal{Z}$ be the covectors that define the facets of Δ and assume that $\Delta^{\alpha_{n+1}}$ is the face that does not contain the origin. Let $v \in R^n$ be the vector defined by the conditions $\alpha_i(v) = l_X(\alpha_i), i = 1, 2 \ldots, n$. Then, for the polytope $X' = X + \{-v\}$, one has $l_{X'}(\alpha_i) = 0 = l_\Delta(\alpha_i)$. Due to Lemma 1, identity (2) implies that $\prod(A \cup \{X'\}) = \sum_i l_{X'}(\alpha_i) = l_{X'}(\alpha_{n+1})$. On the other hand, $\prod(A \cup X') = \prod(A \cup X) = 1$. Therefore, $l_{X'}(\alpha_{n+1}) = 1$ and $X' \subset \Delta$. $\square$

This Lemma and the subsequent one enable to reduce, under appropriate conditions, the statement of Theorem 2 concerning a tuple A to the one concerning a tuple B, which differs from A by one polytope.

Lemma 3 *Let A be an essential n-tuple in $\mathbb{R}^n$ and B be the tuple resulting from A by substitution of $A(i)$ for $A(j)$ for some i, j. If $\prod A = 1$, then $\prod B = 1$.*

Proof. Consider the tuple C resulting from A by substitution of $A(j)$ for $A(i)$. By Aleksandrov–Fenchel inequality (1), we have

$$1 = (\prod A)^2 \geq \prod B \cdot \prod C. \tag{3}$$

Since A is essential, the tuples B and C are linearly independent. Hence, the factors of the right-hand side of (3) are strictly positive and integer, so $\prod B = \prod C = 1$. $\square$

We reduce the dimension n in the proof of Theorem 2 by using the following statement.

Lemma 4 *Let A be an essential n-tuple in $\mathbb{R}^n$ and $B \subset A$ be its subtuple of dimension 1. Then the tuple $C = B \cup \{X\}$ is also essential, where X is the mixed fiber body of $A \setminus B$ in the affine span of $\sum B$.*

Proof. Assume C is not essential: there exists a proper subtuple $C' \subset C$ such that $\dim C' \leq 0$. We have actually $X \in C'$ and $\dim C' = 0$, otherwise the tuple A would be not essential. Consider an arbitrary $Y \in C \setminus C'$. Let $I \subset \sum C'$ be a segment. Consider $A' = A \setminus \{Y\}$, $A_\lambda = A' \cup \{Y + \lambda I\}$, $C_\lambda = (C \setminus \{Y\}) \cup \{Y + \lambda I\}$, $\lambda \geq 0$. We have

$$\prod A_\lambda = \prod C_\lambda = \prod C' \cdot \prod C'' = \prod C = 1,$$

where C'' is the projection of the tuple $C_\lambda \setminus C'$ along the affine span of $\sum C'$, which does not depend on λ. On the other hand, monotony property and linearity imply respectively that

$$\prod A_\lambda \geq \prod (A' \cup \{\lambda I\}) = \lambda \prod (A' \cup \{I\}).$$

Thus we have $\prod(A' \cup \{I\}) = 0$, which contradicts to the fact that A is essential. $\square$

4 Proof of Theorem 2

We construct multi-level induction primarily on n, secondarily on the number k of distinct polytopes in A, thirdly on $g = g(A)$ that is the minimum number among i such that $\dim(A(i) + A(i')) < \dim A(i) + \dim A(i')$ for some $i' < i$ if any, otherwise equal to 1, and finally on the following partial order on the set of n-tuples in $\mathbb{R}^n$. For a tuple A, we consider its multiplicity vector $a = (a(1), a(2), \ldots, a(n))$, where $a(i)$ is the number of polytopes in A that are equal to the polytope $A(i)$ up to translations. A tuple A is said to be richer than a tuple B if the multiplicity vector of A is smaller than that of B in the sense of lexicographic order: $A \rhd B$. Respectively, B is said to be poorer than A: $B \lhd A$. As one can see below, we inductively reduce Theorem statement to poorer tuples.

Base of induction is provided by the cases $n = 1$ and $k = 1$, both are vacuous.

Induction step.

Let $i_1 < i_2 < \ldots < i_k$ be the indices corresponding to the former representatives $A(i_j)$ of k distinct polytopes in the sequence $A(1), A(2), \ldots, A(n)$. Let l be the maximal number in the set $\{0, 1, \ldots, k-1\}$ such that $a(i) = \dim A(i) - 1$ for $1 \leq i \leq i_l$. We split the induction step into several cases as follows.

1. We have $g(A) > 2$. Consider arbitrary $g', g'' \in \{1, 2, \ldots, n\}$ such that $\dim(A(g) + A(g')) < \dim A(g) + \dim A(g')$. Reorder the polytopes of A in such a way that the first two values are $A(g'), A(g'')$ and apply Theorem 2 to the obtained tuple, which holds by induction.

2. There is an n'-subtuple $A' \subset A$ with $n' \leq n - 2$ and $\dim A' = 1$ that contains at least two distinct polytopes. Let $\widetilde{A}$ be the mixed fiber body of $A \setminus A'$ in the affine span L of the polytope $\sum A'$. We have $\prod(A' \cup \widetilde{A}) = \prod A = 1$. Lemma 4 imply also that the tuple $A' \cup \widetilde{A}$ is essential. Since $\dim L < n$, statement of Theorem 2 holds for $A' \cup \widetilde{A}$ by induction, thus there is a volume 1 lattice simplex $\delta \subset L$ containing all the polytopes of A' as well as the polytope $\widetilde{A}$ up to

a translation. Let C' be the tuple formed by $|A'|$ copies of δ. Consider the tuple $C := (A \setminus A') \cup C'$. Due to the monotony property, we have $1 = \prod(A' \cup \widetilde{A}) \leq \prod(C' \cup \widetilde{A}) \leq \mathrm{Vol}(\delta) = 1$ and thus $\prod C = \prod(C' \cup \widetilde{A}) = 1$. As the tuple A is essential, the tuple C is also essential. The number of distinct values of C is less than k since C' has one value δ for at least two distinct values in A', while the values of $A \setminus A'$ do not coincide with those of A' since A is essential. Consequently, the statement of Theorem 2 holds for C by induction and therefore holds for A.

3. The cases 1,2 do not occur and we have $k \geq l+2$. Let B be the n-tuple obtained from A by substitution of the last representative of the polytopes $A(i_k)$ in the sequence $A(1), A(2), \ldots, A(n)$ for $A(i_{l+1})$. Lemma 3 imply that $\prod B = 1$. We also have $g(B) \leq g(A)$.

We claim that B is essential. If not, there exists a proper subtuple $B' \subset B$ containing $a(i_{l+1})+1$ copies of $A(i_{l+1})$ with $\dim B' = 0$. Consider the subtuple A' of A that is obtained from B' by removing one representative of $A(i_{l+1})$. We have $\dim A' = 1$. Since $a_{l+1} < \dim A(i_{l+1})-1$, the tuple A' contains a polytope different from $A(i_{l+1})$ and therefore the case 2 occur.

The number of distinct values of B is less or equal k, $g(B) \leq g(A) \leq 2$, while $B \lhd A$ since the multiplicities $b(i)$ of the polytopes $B(i)$ in B coincide with $a(i)$ for $i < i_{l+1}$ and $b(i_{l+1}) = a(i_{l+1}) + 1$. Therefore, statement of Theorem 2 holds for B by induction: there is a volume 1 lattice simplex $\Delta \in \mathbb{R}^n$ containing all $B(i)$ and thus also all the polytopes $A(i)$, possibly except for $A(i_k)$, up to translations. Applying Lemma 2, we conclude that Δ contains also $A(i_k)$ up to a translation, which completes the proof.

4. The cases 1,2 do not occur and $k = l+1$. This one is the most difficult case and we need two more lemmas to cope with it. The first one is a technical tool and the second one covers the statement of Theorem 2 for a distinguished special case.

Definition. Polytopes $A(1), A(2), \ldots, A(k) \subset \mathbb{R}^n$ are said to be *transversal*, if $\dim \sum A(i) = \sum \dim A(i)$. A polytope A is said to be *transversal to* a set of polytopes $B_1, B_2, \ldots, B_k$, if the polytopes $A, B_1, \ldots, B_k$ are transversal.

Lemma 5 *Assume an n-tuple A in $\mathbb{R}^n$ contains transversal volume 1 lattice simplices $\delta_1, \delta_2, \ldots, \delta_p \subset \mathbb{R}^n$, each δ_j with multiplicity $a(j) = \dim \delta_j - 1$. Assume A contains also a polytope X, which is transversal to each of δ_j and though not transversal to the set $\delta_1, \delta_2, \ldots, \delta_p$. Assume $\prod A = 1$ and $\prod(A' \cup \{\delta_j\}) = 1$, $j = 1, 2, \ldots, p$, where $A' := A \setminus \{X\}$. There exist s_1, s_2 and a volume 1 lattice simplex Δ adjacent to $\delta_{s_1}, \delta_{s_2}$ up to translations such that $\dim \Delta - \dim \delta_{s_1} - \dim \delta_{s_2}$ equals 0 or 1 and $\prod(A' \cup \Delta) = 1$.*

Proof. Let $\Psi \subset \mathcal{Z}$ be the set of covectors α such that $\prod(A')^\alpha > 0$. We use the following

Proposition 3 *For each $j = 1, 2, \ldots, p$:*

1. *There are covectors $\alpha_{j1}, \ldots, \alpha_{jl_j} \in \Psi$, where $l_j = \dim \delta_j + 1$, such that $\prod(A')^{\alpha_{js}} = 1$ and $\delta_j^{\alpha_{js}}$, $s = 1, 2, \ldots, l_j$, are the l_j facets of δ_j.*
2. *For each covector $\alpha \in \Psi \setminus \{\alpha_{j1}, \alpha_{j1}, \ldots, \alpha_{jl_j}\}$, we have $\delta_j^\alpha = \delta_j$.*

Proof. Let $B_j \subset A$ be the subtuple consisting of $a(j)$ copies of δ_j. Consider the projection C_j of the tuple $A' \setminus B_j$ along the affine span L_j of the simplex δ_j. We have $\prod C_j = \prod C_j \cdot \mathrm{Vol}(\delta_j) = \prod(A' \cup \{\delta_j\}) = 1$. Let $v_1, v_2, \ldots, v_{l_j}$ be the vertices of the simplex δ_j. For each edge $e_{st} = \{v_s, v_t\}, s \neq t$, we have

$$\prod(A' \cup \{e_{st}\}) = \prod(B_j \cup \{e_{st}\}) \cdot \prod C_j = 1. \tag{4}$$

On the other hand, equation (2) implies:

$$\prod(A' \cup \{e_{st}\}) = \sum_{\alpha \in \Psi} \left(\max(\alpha(v_s), \alpha(v_t)) \cdot \prod(A')^{\alpha} \right). \tag{5}$$

For a coordinate system in $\mathbb{R}^n$ centered in v_t, the summands of the right-hand side of (5) are non-negative and integer. Therefore, the equations (4) and (5) imply: there is a covector $\alpha_{st} \in \Psi$ with $\alpha_{st}(v_s) = 1$, $\prod(A')^{\alpha_{st}} = 1$ and for any other covector $\alpha \in \Psi$, we have $\alpha(v_s) = 0$. Observe that

$$1 = \prod(A' \cup \{\delta_j\}) = \sum_{\alpha \in \Psi} \left(l_{\delta_j}(\alpha) \cdot \prod(A')^{\alpha} \right), \tag{6}$$

thus, for the coordinate system centered in v_t, there is exactly one covector $\alpha_{jt} \in \Psi$ such that $\max(\alpha_{jt}|_{\delta_j}) > 0$. In particular, $\alpha_{st} = \alpha_{jt}$ not depending on s and thus we have $\alpha_{jt}(v_t) = 0$ and $\alpha_{jt}(v_i) = 1$ for each $i \neq t$. Therefore, $\delta_j^{\alpha_{jt}}$ is the facet of δ_j that do not contain v_t. Equation 6 also implies that $\prod(A')^{\alpha_{jt}} = 1$ and this completes part 1. For part 2, we imply the same arguments by utilizing (6). $\square$

We continue the proof of Lemma 5 by the two following cases.

1. Assume we have $\alpha_{us_1} = \alpha_{vs_2} = \alpha_0$ for some $u \neq v$ and s_1, s_2. Assume for simplicity that the vertices of the polytopes δ_u, δ_v not belonging to $\delta_u^{\alpha_{us_1}}, \delta_v^{\alpha_{vs_2}}$ coincide with the origin (it is actually true up to translations). Consider $\Delta = \langle \delta_u \cup \delta_v \rangle$. In accordance with Proposition 3, equation 2 implies that $\prod(A' \cup \Delta) = \sum_{\alpha \in \Psi} l_\Delta(\alpha) \prod(A')^{\alpha} = l_\Delta(\alpha_0) \cdot 1 = 1$. In this case, we have $\dim \Delta - \dim \delta_u - \dim \delta_v = 0$.

2. Now assume the contrary: we have $\sum_{j=1}^{p} l_j$ different covectors α_{js}. Assume for simplicity that each of δ_j, $j = 1, 2, \ldots, p$, contains the origin. Since $\dim(X + \sum_{j=1}^{p} \delta_j) < \dim X + \dim \sum_{j=1}^{p} \delta_j$, there exists a non-degenerate segment $[x, y] \subset X$ such that $z = x - y \in \oplus_{i=1}^{p} L_j$, that is, $z = \sum z_j$, where z_j belongs to the affine span L_j of δ_j. There are at least two values u, v of the index j such that $z_j \neq 0$, otherwise, the polytope X would be transversal to those δ_j corresponding to the unique $z_j \neq 0$. Due to Proposition 3, part 2, we have $\alpha_{j_1 s}(z_{j_2}) = 0$ for $j_1 \neq j_2$ and thus $\alpha_{js}(z) = \alpha_{js}(z_j)$. There exist s_1, s_2 such that $\alpha_{us_1}(z_u) > 0$ and $\alpha_{vs_2}(z_v) > 0$. This means that $\alpha_{us_1}(x) > \alpha_{us_1}(y)$, $\alpha_{vs_2}(x) > \alpha_{vs_2}(y)$, and thus the covectors $\alpha_1 = \alpha_{us_1}, \alpha_2 = \alpha_{vs_2}$ are not constant on the polytope X. For each of the two covectors, the range of values on X is a unit segment because of the following equation:

$$\sum_{\alpha \in \Psi} \left(l_X(\alpha) \cdot \prod(A')^{\alpha} \right) = \prod A. \tag{7}$$

First, assume there is a vertex $x \in X$ with $\alpha_1(x) = \min(\alpha_1|_X), \alpha_2(x) = \min(\alpha_2|_X)$. W.l.o.g., assume that x coincides with the origin. The left-hand side summands of equation (7) are non-negative, while those for $\alpha = \alpha_1, \alpha_2$ are positive and integer. Therefore, $\prod A \geq 2$, which contradicts to the conditions of Lemma 5.

Thus it follows such a vertex x does not exist. In this case, there exist two different vertices $x_1, x_2 \in X$ such that $\alpha_1(x_2) = \alpha_1(x_1) + 1, \alpha_2(x_1) = \alpha_2(x_2) + 1$. Denote $x = x_2 - x_1$. We claim that $\alpha(x) = 0$ for $\alpha \in \Psi, \alpha \neq \alpha_1, \alpha_2$. If not, there exist $\alpha_0 \in \Psi, \alpha_0 \neq \alpha_1, \alpha_2$ with $\alpha_0(x_2) > \alpha_0(x_1)$ (or $\alpha_0(x_1) > \alpha_0(x_2)$). Translate the polytope X in such a way that x_1 coincides with the origin (or x_2 coincides with the origin). The left-hand side summands of equation (7) become non-negative, while those corresponding to $\alpha = \alpha_1, \alpha_0$ (or $\alpha = \alpha_2, \alpha_0$) are strictly positive. Therefore, $\prod A \geq 2$, which is a contradiction.

Translate the polytopes δ_u, δ_v in such a way that their vertices not belonging to $\delta_u^{\alpha_1}, \delta_v^{\alpha_2}$ coincide with the origin. Next, translate the polytope δ_v by adding vector x. Consider the simplex $\Delta = \langle \delta_u \cup \delta_v \rangle$. We have $\max(\alpha_{us}|_\Delta) = 0, s \neq s_1, \max(\alpha_{vs}|_\Delta) = 0, s \neq s_2, \max(\alpha_{us_1}|_\Delta) = 1$, $\max(\alpha_{vs_2}|_\Delta) = 0$, and $\max(\alpha|_\Delta) = 0$ for other covectors $\alpha \in \Psi$. Therefore we have $\prod(A' \cup \Delta) = \sum_{\alpha \in \Psi} (l_\Delta(\alpha) \cdot \prod(A')^\alpha) = 1$. In this case, $\dim \Delta - \dim \delta_u - \dim \delta_v = 1$. $\square$

Lemma 6 *Let $X \subset \mathbb{R}^n$ be a lattice polytope and $\delta \subset \mathbb{R}^n$ be a volume 1 lattice simplex, $\dim \delta = d < n$. Let A be the tuple consisting of $(d-1)$ copies of δ and $n-d+1$ copies of X. Assume $B = (A \setminus \{X\}) \cup \{\delta\}$. If $\prod A = \prod B = 1$, there exists a volume 1 lattice simplex Δ that contains the polytopes δ, X up to translations.*

Proof. Proposition 1 implies that the tuple B is linearly independent and thus the projection Y of the polytope X along the affine span of the simplex δ has dimension $n-d$. Therefore, there exists a volume 1 lattice simplex $\delta' \subset X$ transversal to δ such that $\dim \delta' = n-d$. Since A is linearly independent, we have $\dim X \geq n-d+1$ and therefore X is not contained in the affine span of δ'. Thus there exists a lattice segment $X' \subset X$, which is transversal to δ, as well as to δ'. Consider the tuple C consisting of $d-1$ copies of δ, $n-d-1$ copies of δ', and one copy of each of X and X'. Assume $C' = C \setminus \{X'\}$, $C_1 = C' \cup \{\delta'\}$, $C_2 = C' \cup \{\delta\}$. The polytopes C, C_1, C_2 are transversal. Since the monotony property implies that $\prod C_1, \prod C \leq \prod A = 1$, $\prod C_2 \leq \prod B = 1$, we actually observe $\prod C = \prod C_1 = \prod C_2 = 1$. Applying Lemma 5 to the tuple C, we obtain a volume 1 lattice full-dimensional simplex Δ in $\mathbb{R}^n$ such that $\prod(C' \cup \{\Delta\}) = 1$ and the simplices δ, δ' are contained in Δ up to translations. Note that the tuple $C' \cup \{\Delta\}$ is essential and thus Lemma 2 imply that the polytope $X \in C'$ is also contained in Δ up to a translation. $\square$

Now we return to the case 4 of induction step. Assume $\dim A(i_{j_0}) = n$ for some $j_0 \in \{1, 2 \ldots, l\}$. Then $a(i_{j_0}) = n-1$ and we have actually $k = 2$. Lemma 3 implies that the volume of $A(i_{j_0})$ equals 1 and thus the polytope $A(i_{j_0})$ is a volume 1 lattice simplex Δ. Lemma 2 implies also that the only polytope of A different from $A(i_{j_0})$ is also contained in Δ up to a translation, which completes the proof.

In what follows, we assume that $\dim A(i_j) < n$ for $j = 1, 2, \ldots, l$. Let us show that each of the polytopes $A(i_j)$, $j = 1, 2 \ldots, l$, is a simplex. Since the affine span L_j of the polytope $A(i_j)$ has dimension lesser n, Theorem 2 holds there by induction. Let B_j be the tuple consisting of $a(i_j)$ copies of $A(i_j)$. Let $\widetilde{A}_j \subset L_j$ be the mixed fiber body of the set of the other polytopes of A. Applying Theorem 2 to the tuple $B_j \cup \{\widetilde{A}_j\}$, we get a volume 1 lattice simplex $\delta_j \in L_j$, which contains $A(i_j)$. Since $\dim A(i_j) = \dim \delta_j$, we get $A(i_j) = \delta_j$.

Note that $l = k - 1 > 0$ and, therefore, $A(i_1)$ is a simplex. We consider the two following subcases below:

A). We have $g = 2$. For $k = 2$, apply Lemma 6. Assume $k > 2$. In this case, $l \geq 2$ and thus the subtuple consisting of $a(i_1)$ copies of $A(i_1)$ and $a(i_2)$ copies of $A(i_2)$ has dimension 1. If we had also $a(i_1) + a(i_2) < n - 1$, then the case 2 would occur. Thereby, $a(i_1) + a(i_2) = n - 1$ and $k = 3$. We apply Lemma 6 to the tuple $B := (A \setminus \{A(i_3)\}) \cup \{A(i_2)\}$ and obtain a volume 1 lattice simplex Δ, which contains the polytopes of B up to translations. Since the tuple A differs from B in a single polytope, Lemma 2 completes the proof.

B). We have $g = 1$. Let $p \in \{1, 2, \ldots, k\}$ be the maximal number such that the polytopes $A(i_1), A(i_2), \ldots, A(i_p)$ are transversal. We claim that $p \leq l = k - 1$. In fact,

$$n = \sum_{j=1}^{k} a(i_j) < \sum_{j=1}^{k} \dim A(i_j).$$

Taking into account that $n = \dim \sum_{j=1}^{k} A(i_j)$, we get $\dim \sum_{j=1}^{k} A(i_j) < \sum_{j=1}^{k} \dim A(i_j)$, thus $p \neq k$.

Consider the polytope $X = A(i_{p+1})$ and the $(n-1)$-tuple $A' := A \setminus \{X\}$. Since $g = 1$, the polytope X is transversal to each of $A(i_j), j \leq p$. Lemma 5 provides a volume 1 lattice simplex Δ such that $\prod(A' \cup \{\Delta\}) = 1$, $A(i_u), A(i_v)$ are contained in Δ up to translations for some $u, v \leq p$ and $\dim \Delta - \dim A(i_u) - \dim A(i_v) = d$, which equals 0 or 1. We claim that $B = A' \cup \{\Delta\}$ is essential. If not, there is a subtuple $C \subset A$ such that $A(i_u), A(i_v) \in C$, $\dim C = 1$, $|C| \leq n-2$, and hence the case 2 occurs.

Let $B' \subset B$ be the subtuple consisting of $a(i_u)$ copies of $A(i_u)$, $a(i_v)$ copies of $A(i_v)$ and the polytope Δ. The polytopes of B' are contained in the volume 1 lattice simplex Δ and we have $\dim B' = 1$. Consider the mixed fiber body $\widetilde{B}$ of $B \setminus B'$ in the affine span of Δ. Lemma 4 implies that the tuple $B'' = B' \cup \{\widetilde{B}\}$ is essential. We have also $\prod B'' = \prod B = 1$, thus Lemma 2 implies that $\widetilde{B}$ is contained in Δ up to a translation. Let C be the tuple formed by $|B'|$ copies of Δ. The tuple $B_1 = (B \setminus B') \cup C$ is essential since the tuple B is. We have also $\prod B_1 = \prod C \cup \{\widetilde{B}\} = 1$. The number of distinct values of B_1 is lesser than k. Therefore, Theorem 2 holds for B_1 by induction and thus holds for B. Since A differs from B in a single polytope, Lemma 2 completes the proof.

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